

An inexact Picard iteration method for absolute value equation

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Abstract Recently, a class of inexact Picard iteration method for solving the absolute value equation: $Ax - |x| = b$ have been proposed in [Optim Lett 8:2191-2202,2014]. To further improve the performance of Picard iteration method, a new inexact Picard iteration method is proposed to solve the absolute value equation. The sufficient conditions for the convergence of the proposed method for the absolute value equation is given. Some numerical experiments are given to demonstrate the effectiveness of the new method.

Keywords Absolute value equation · Picard iteration method · inexact Picard iteration method · Convergence

1 Introduction

In this study, we consider the iteration method for solving the absolute value equation (AVE) of the form

$$Ax - |x| = b, \quad (1)$$

where $A \in \mathbb{C}^{n \times n}$, $x, b \in \mathbb{C}^n$, and $|x|$ denotes the component-wise absolute value of the vector x , i.e., $|x| = (|x_1|, \dots, |x_n|)^T$. The AVE (1) is a special case of the generalized absolute value equation of the type

$$Ax + B|x| = b, \quad (2)$$

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where $B \in \mathbb{C}^{n \times n}$. The generalized absolute value equation (2) was introduced in [12] and investigated in a more general context [8, 9, 11]. The absolute value equation (1) or (2) arises in a variety of optimization problems, e.g. linear complementarity problem, linear programming or convex quadratic programming problems; see for example [4, 8, 9, 11, 12].

For AVE (1), one can deduce that all singular values of A exceeding 1 implies existence of a unique solution for every right-hand side b [9]. When AVE (1) has the unique solution, how to find the solution of (1) is a main research topic. In recent years, a large variety of methods for solving AVE (1) can be found in the literature [3, 6, 7, 10, 13, 15, 16]. Among these methods, Picard-type methods capture one's attention. Rohn et al. in [13] proposed a class of method to solve AVE (1), in practice their method is reduced to the well known Picard iteration method

$$x^{(k+1)} = A^{-1} \left(|x^{(k)}| + b \right), \quad k = 0, 1, 2, \dots, \quad (3)$$

where $x^{(0)} = A^{-1}b$ is the initial guess. From (3), we can see that there is a linear system with the constant coefficient matrix A should be solved in each iteration of the Picard method. To improve the performance of the Picard method, the linear system with matrix A should be solved by inner iteration, this leads to inexact Picard iteration method. As an example, Salkuyeh suggested that using Hermitian and skew-Hermitian splitting iteration (HSS) method [1] to approximation the solution of the linear system with A at each Picard iteration, and proposed the Picard-HSS method for solving AVE(1) [15]. In fact, the Picard-HSS method has been proposed originally by Bai and Yang for weakly nonlinear systems in [2]. The sufficient conditions to guarantee the convergence of the Picard-HSS method and some numerical experiments are given to show the effectiveness of the method for solving AVE(1) in [15].

Bear in mind that there are two linear subsystem need to be solved at each step of the inner HSS iteration of the Picard-HSS method [2, 15], one is the linear subsystem with shift Hermitian coefficient matrix and the other is the ones with shift skew-Hermitian coefficient matrix. The solution of linear subsystem with shift Hermitian coefficient matrix can be easily obtained by CG method, however, the solution of linear subsystem with shift skew-Hermitian coefficient matrix is not easy to obtain, in some cases, its solution is as difficult as that of the original linear system. To avoid solving a linear subsystem with shift coefficient skew-Hermitian in the inner iteration of the inexact Picard method, we use the single-step HSS method [5] to approximate the solution of the linear system with coefficient matrix A and present a new inexact Picard method, abbreviated as Picard-SHSS iteration method, in this paper.

The rest of this paper is organized as follows. In Section 2, after review some notes and the single-step HSS iteration method, the Picard-SHSS iteration method for solving AVE (1) is described. And then the convergence properties of the Picard-SHSS iteration method is studied. Numerical experiments are presented in Section 3, to show the feasibility and effectiveness of the Picard-SHSS method.

2 The Picard-SHSS method

For convenience, some notations, definitions and results that will be used in the following parts are given below. For a matrix $A \in \mathbb{C}^{n \times n}$, A^* represents the conjugate transpose of A , and $\rho(A)$ denotes the spectral radius of A . A is said to be non-Hermitian positive definite if its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite, i.e., $x^* H x > 0$ for any $x \in \mathbb{C}^n \setminus \{0\}$.

Let $A \in \mathbb{C}^{n \times n}$ be a non-Hermitian positive definite matrix, and $A = H + S$ be its Hermitian and skew-Hermitian splitting (HSS) with

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*).$$

Based on the HSS of A , Bai et al. [1] presented the HSS iteration method to solve non-Hermitian positive definite system of linear equations $Ax = q$. There are two linear subsystem need to be solved at each step of the HSS iteration method, one is the linear subsystem with shift Hermitian coefficient matrix $\alpha I + H$ and the other is the ones with shift skew-Hermitian coefficient matrix $\alpha I + S$ for any positive constant α and identity matrix I ; see [1] for more details. The challenges of the HSS iteration method lies in solving the linear subsystem with $\alpha I + S$, which is as difficult as that of the original linear system in some cases. To avoid solving a linear subsystem with $\alpha I + S$ in the HSS iteration method, the single-step HSS method is proposed recently [5]. The iteration scheme of the single-step HSS method used for solving system of linear equations $Ax = q$ can be written equivalently as

$$(\alpha I + H)x^{(k+1)} = (\alpha I - S)x^{(k)} + q, \quad (4)$$

here α is a positive iteration parameter. It has been proved that, under a loose restriction on the iteration parameter α , the single-step HSS method is convergent to the unique solution of the linear system $Ax = q$ for any initial guess $x^{(0)} \in \mathbb{C}^n$; see [5].

Recalling that the Picard iterative method for solving AVE (1) is a fixed-point iterative method of the form

$$Ax^{(k+1)} = |x^{(k)}| + b, \quad k = 0, 1, 2, \dots \quad (5)$$

We assume that the matrix A is non-Hermitian positive definite. In this case, the next iterate of $x^{(k+1)}$ can be approximately computed by the single-step HSS iteration by making use of the splitting $A = M(\alpha) - N(\alpha)$ as following (see [2])

$$\begin{aligned} M(\alpha)x^{(k,l+1)} &= N(\alpha)x^{(k,l)} + |x^{(k)}| + b, \\ l &= 0, 1, \dots, l_k - 1, \\ k &= 0, 1, 2, \dots, \end{aligned} \quad (6)$$

where $M(\alpha) = \alpha I + H$ and $N(\alpha) = \alpha I - S$, α is a positive constant, $\{l_k\}_{k=0}^{\infty}$ a prescribed sequence of positive integers, and $x^{(k,0)} = x^{(k)}$ is the starting point of the inner single-step HSS iteration at k -th outer Picard iteration. This leads to the inexact Picard iteration method, called Picard-SHSS iteration method, for solving the system (1) which can be summarized as following

The Picard-SHSS iteration method: Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be the Hermitian and skew-Hermitian

parts of A respectively. Given an initial guess $x^{(0)} \in \mathbb{C}^n$ and a sequence $\{l_k\}_{k=0}^\infty$ of positive integers, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

- (a). Set $x^{(k,0)} = x^{(k)}$;
- (b). For $l = 0, 1, \dots, l_k - 1$, solve the following linear system to obtain $x^{(k,l+1)}$:

$$(\alpha I + H)x^{(k,l+1)} = (\alpha I - S)x^{(k,l)} + |x^{(k)}| + q,$$

where α is a positive constant and I is the identity matrix;

- (c). Set $x^{(k+1)} = x^{(k,l_k)}$.

Compared with the Picard-HSS iteration method studied in [15], a linear subsystem with $\alpha I + S$ is avoided in the inner iteration of the Picard-SHSS iteration method. The involved linear subsystem with $\alpha I + H$ of the Picard-SHSS iteration method can be efficiently solved exactly by a sparse Cholesky factorization, or inexactly by a preconditioned Conjugate Gradient method [14].

The next theorem provides sufficient conditions for the convergence of the Picard-SHSS method to solve the AVE (1).

Theorem 1 Let $A \in \mathbb{C}^{n \times n}$ be a non-Hermitian positive definite matrix and $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively. Let α be a constant number such that $\alpha > \max\left\{0, \frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}}\right\}$, where λ_{\min} is the smallest eigenvalue of H and σ_{\max} is the largest singular-value of S . Let also $\eta = \|A^{-1}\|_2 < 1$. Then the AVE (1) has a unique solution x^* , and for any initial guess $x^{(0)} \in \mathbb{C}^n$ and any sequence of positive integers l_k , $k = 0, 1, \dots$, the iteration sequence $\{x^{(k)}\}_{k=0}^\infty$ produced by the Picard-SHSS iteration method converges to x^* provided that $l = \liminf_{k \rightarrow \infty} l_k \geq N$, where N is a natural number satisfying

$$\|T(\alpha)^s\|_2 < \frac{1 - \eta}{1 + \eta} \quad \forall s \geq N.$$

Proof Let $T(\alpha) = M(\alpha)^{-1}N(\alpha)$, based on the iteration scheme (6), we can express the $(k+1)$ th iterate $x^{(k+1)}$ of the Picard-SHSS iteration method as

$$x^{(k+1)} = T(\alpha)^{l_k} x^{(k)} + \sum_{j=0}^{l_k-1} T(\alpha)^j M(\alpha)^{-1}(|x^{(k)}| + b), \quad k = 0, 1, 2, \dots \quad (7)$$

Note that $\eta < 1$, then AVE (1) has a unique solution $x^* \in \mathbb{C}^n$ [9] such that

$$x^* = T(\alpha)^{l_k} x^* + \sum_{j=0}^{l_k-1} T(\alpha)^j M(\alpha)^{-1}(|x^*| + b), \quad k = 0, 1, 2, \dots \quad (8)$$

By subtracting (8) from (7) we have

$$x^{(k+1)} - x^* = T(\alpha)^{l_k} (x^{(k)} - x^*) + \sum_{j=0}^{l_k-1} T(\alpha)^j M(\alpha)^{-1}(|x^{(k)}| - |x^*|). \quad (9)$$

It follows from [5, Theorem 2.1] that $\rho(T(\alpha)) < 1$ when α satisfying $\alpha > \max\left\{0, \frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}}\right\}$. In this case, some calculations yield

$$\sum_{j=0}^{l_k-1} T(\alpha)^j M(\alpha)^{-1} = \left(I - T(\alpha)^{l_k}\right) A^{-1}.$$

Now (9) becomes

$$\begin{aligned} x^{(k+1)} - x^* &= T(\alpha)^{l_k} (x^{(k)} - x^*) + \left(I - T(\alpha)^{l_k}\right) A^{-1} (|x^{(k)}| - x^*) \\ &= T(\alpha)^{l_k} \left[(x^{(k)} - x^*) - A^{-1} (|x^{(k)}| - x^*) \right] + A^{-1} (|x^{(k)}| - x^*). \end{aligned}$$

Note that $\| |x| - |y| \|_2 \leq \|x - y\|_2$ for any $x, y \in \mathbb{C}^n$, it then follows that

$$\|x^{(k+1)} - x^*\|_2 \leq \left(\|T(\alpha)^{l_k}\|_2 (1 + \eta) + \eta \right) \|x^{(k)} - x^*\|_2.$$

The condition of $\rho(T(\alpha)) < 1$ when α satisfying $\alpha > \max\left\{0, \frac{\sigma_{\max}^2 - \lambda_{\min}^2}{2\lambda_{\min}}\right\}$ ensure that $T(\alpha)$ tend to 0 as s tend to infinity. Therefore, there is a natural number N such that

$$\|T(\alpha)^s\|_2 < \varepsilon := \frac{1 - \eta}{1 + \eta} \quad \forall s \geq N.$$

Now, if we let $l = \liminf_{k \rightarrow \infty} l_k \geq N$, then $\|x^{(k+1)} - x^*\|_2 < \|x^{(k)} - x^*\|_2$, hence the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ produced by the Picard-SHSS iteration method converges to x^* . \square

In actual computation, the residual-updating form of the Picard-SHSS iteration method is more convenient, which can be written as following.

The Picard-SHSS iteration method (residual-updating variant): Let $A \in \mathbb{C}^{n \times n}$ be no-Hermitian positive definite, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be the Hermitian and skew-Hermitian parts of A respectively. Given an initial guess $x^{(0)} \in \mathbb{C}^n$ and a sequence $\{l_k\}_{k=0}^{\infty}$ of positive integers, compute $x^{(k+1)}$ for $k = 0, 1, 2, \dots$ using the following iteration scheme until $\{x^{(k)}\}$ satisfies the stopping criterion:

- (a). Set $s^{(k,0)} = 0$ and $b^{(k)} = |x^{(k)}| + b - Ax^{(k)}$;
- (b). For $l = 0, 1, \dots, l_k - 1$, solve the following linear system to obtain $s^{(k,l+1)}$:

$$(\alpha I + H)s^{(k,l+1)} = (\alpha I - S)s^{(k,l)} + b^{(k)},$$

where α is a positive constant and I is the identity matrix;

- (c). Set $x^{(k+1)} = x^{(k)} + s^{(k,l_k)}$.

3 Numerical experiments

In this section we give some numerical experiments to show the effectiveness of the Picard-SHSS iteration method to solve AVE (1), to do this, the numerical properties of the Picard-HSS and Picard-SHSS methods are examined and compared experimentally by a suit of test problems. We use the residual-updating versions of the Picard-HSS iteration method [15] and Picard-SHSS iteration method.

All the numerical experiments presented in this section have been computed in double precision using some MATLAB R2012b on Intel(R) Core(TM) i5-2400 CPU 3.10 GHz and 4.00 GB of RAM. All runs are started from the initial zero vector and terminated if the current relative residual satisfies

$$\text{RES} := \frac{\|Ax^{(k)} - |x^{(k)}| - b\|_2}{\|b\|_2} \leq 10^{-7},$$

where $x^{(k)}$ is the computed solution by each of the methods at iteration k , and a maximum number of the iterations 500 is used. In addition, the stopping criterion for the inner iterations of the Picard-HSS and Picard-SHSS methods are set to be

$$\frac{\|b^{(k)} - A_S^{(k,l)}\|_2}{\|b^{(k)}\|_2} \leq 0.01,$$

and a maximum number of the iterations 10 ($l_k = 10$, $k = 0, 1, 2, \dots$) for inner iterations are used. The right-hand side vector of AVE (1) is taken such a way that the vector $x = (x_1, x_2, \dots, x_n)^T$ with

$$x_i = (-1)^i i, \quad i = 1, 2, \dots, n$$

be the exact solution.

The optimal parameters employed in the Picard-HSS and Picard-SHSS iteration methods are chosen to be the experimentally found optimal ones, which result in the least number of iteration steps of iteration methods.

The coefficient matrix A of AVE (1) is given by

$$A = T_x \otimes I_m + I_m \otimes T_y + pI_n, \quad (10)$$

where I_m and I_n are the identity matrices of order m and n with $n = m^2$, \otimes means the Kronecker product, T_x and T_y are tridiagonal matrices

$$T_x = \text{tridiag}(t_2, t_1, t_3)_{m \times m} \quad \text{and} \quad T_y = \text{tridiag}(t_2, 0, t_3)_{m \times m}$$

with $t_1 = 4$, $t_2 = -1 - Re$, $t_3 = -1 + Re$. Here $Re = (qh)/2$ and $h = 1/(m+1)$ are the mesh Reynolds number and the equidistant step size, respectively, and q is a positive constant. In fact, the matrix A arising from the finite difference approximation the two-dimensional convection-diffusion equation

$$\begin{cases} -(u_{xx} + u_{yy}) + q(u_x + u_y) + pu = f(x, y), & (x, y) \in \Omega, \\ u(x, y) = 0, & (x, y) \in \partial\Omega, \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$, $\partial\Omega$ is its boundary, q is a positive constant used to measure the magnitude of the diffusive term and p is a real number. If we use the five-point finite difference scheme to the diffusive terms and the central difference

scheme to the convective terms, then we obtained the matrix A . It is easy to find that for every nonnegative number q the matrix A is in general non-symmetric positive definite [15].

In our numerical experiments, the matrix A in AVE (1) is defined by (10) with different values of q ($q = 0, 1, 10$, and 100) and different values of p ($p = 0$ and -1). In Table 2 and Table 4, we present the numerical results with respect to the Picard-HSS and Picard-SHSS iteration methods, the experimentally optimal parameters used in the Picard-HSS and Picard-SHSS iteration methods are those given in Table 1 and Table 3. We give the elapsed CPU time in seconds for the convergence (denoted by CPU), the number of iterations for the convergence (denoted by IT) and the relative residuals (denoted by RES).

From the Table 2 and Table 4, we see that both the Picard-HSS and Picard-SHSS iteration methods can successfully produced approximate solution to the AVE (1) for all of the problem-scales $n = m^2$ and the convective measurements q . For the convergent cases, the CPU time also increases rapidly with the increasing of the problem-scale for all tested iteration methods. Moreover, numerical results in the two tables show that the Picard-SHSS iteration method perform better than the Picard-HSS iteration method in most cases as the former one cost the least CPU time to achieve stopping criterion except the case of $q = 100$ and $n = 100, 400$. In addition, for $p = -1$, the Picard-SHSS iteration method costs the least number of iteration steps and CPU time to achieve stopping criterion. In summary, the Picard-SHSS iteration method is useful and effective for solving the NP-hard AVE (1).

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Table 1 The optimal parameters for Picard-HSS and Picard-SHSS methods ($p = 0$)

Optimal parameters		$m = 10$	$m = 20$	$m = 40$	$m = 80$
$q = 0$	Picard-HSS	11.69	12.6	13.4	13
	Picard-SHSS	5.745	6.5	6.4	6.6
$q = 1$	Picard-HSS	12.01	13.6	14	13
	Picard-SHSS	5.926	6.6	6.75	6.6
$q = 10$	Picard-HSS	6.76	10.99	13.43	15.8
	Picard-SHSS	3.594	5.52	6.63	8.0
$q = 100$	Picard-HSS	23.3	23.1	8.7	9.2
	Picard-SHSS	81.5	26.4	4.99	4.57

Table 2 Numerical results for different values of m and q ($p = 0$)

Methods			$m = 10$	$m = 20$	$m = 40$	$m = 80$
$q = 0$	Picard-HSS	IT	36	32	30	28
		CPU	0.0280	0.0371	0.1266	0.9486
		RES	$9.8815e - 8$	$9.3643e - 8$	$9.9900e - 8$	$9.9043e - 8$
	Picard-SHSS	IT	37	32	30	29
		CPU	0.0137	0.0245	0.1075	0.9060
		RES	$9.4432e - 8$	$9.6857e - 8$	$9.7569e - 8$	$9.9256e - 8$
$q = 1$	Picard-HSS	IT	35	32	31	28
		CPU	0.0220	0.0457	0.2377	1.9978
		RES	$9.1372e - 8$	$9.8614e - 8$	$9.3177e - 8$	$9.7012e - 8$
	Picard-SHSS	IT	36	32	31	29
		CPU	0.0136	0.0273	0.1292	1.1940
		RES	$9.8685e - 8$	$9.4248e - 8$	$9.4655e - 8$	$9.6233e - 8$
$q = 10$	Picard-HSS	IT	29	66	33	36
		CPU	0.0163	0.0939	0.2636	2.5265
		RES	$9.5635e - 8$	$9.9755e - 8$	$9.8395e - 8$	$9.9024e - 8$
	Picard-SHSS	IT	29	63	33	37
		CPU	0.0105	0.0528	0.1372	1.4083
		RES	$9.6650e - 8$	$9.8439e - 8$	$9.7901e - 8$	$9.8170e - 8$
$q = 100$	Picard-HSS	IT	11	17	35	146
		CPU	0.0117	0.0342	0.2826	9.6469
		RES	$9.8396e - 8$	$9.9503e - 8$	$9.3177e - 8$	$9.7165e - 8$
	Picard-SHSS	IT	44	25	35	140
		CPU	0.0161	0.0246	0.1471	4.9182
		RES	$9.8783e - 8$	$9.8253e - 8$	$9.8972e - 8$	$9.7299e - 8$

Table 3 The optimal parameters for Picard-HSS and Picard-SHSS methods ($p = -1$)

Optimal parameters		$m = 10$	$m = 20$	$m = 40$	$m = 80$
$q = 0$	Picard-HSS	13.8	11.2	10.36	10.1
	Picard-SHSS	6.96	5.7	5.21	5.1
$q = 1$	Picard-HSS	14	11.29	10.4	11
	Picard-SHSS	7.05	5.72	5.2	5.1
$q = 10$	Picard-HSS	20.55	13.6	11.1	11
	Picard-PHSS	11.2	7.0	5.65	5.2
$q = 100$	Picard-HSS	27.2	22	11.1	21
	Picard-SHSS	108	33.1	11.57	10.65

Table 4 Numerical results for different values of m and q ($p = -1$)

Methods			$m = 10$	$m = 20$	$m = 40$	$m = 80$
$q = 0$	Picard-HSS	IT	24	62	206	769
		CPU	0.0186	0.0944	0.8224	25.7551
		RES	$9.6164e - 8$	$9.5608e - 8$	$9.8436e - 8$	$9.8404e - 8$
	Picard-SHSS	IT	21	52	166	614
		CPU	0.0099	0.0444	0.5608	18.5361
		RES	$9.6537e - 8$	$9.7425e - 8$	$9.9905e - 8$	$9.8795e - 8$
$q = 1$	Picard-HSS	IT	24	61	203	908
		CPU	0.0193	0.0877	1.4320	57.8838
		RES	$9.7842e - 8$	$9.9013e - 8$	$9.3292e - 8$	$9.9324e - 8$
	Picard-SHSS	IT	21	51	166	598
		CPU	0.0102	0.0421	0.6038	18.9311
		RES	$9.5087e - 8$	$9.5556e - 8$	$9.5308e - 8$	$9.8338e - 8$
$q = 10$	Picard-HSS	IT	14	27	74	297
		CPU	0.0175	0.0519	0.5532	18.6613
		RES	$9.8754e - 8$	$9.1215e - 8$	$9.8943e - 8$	$9.6129e - 8$
	Picard-SHSS	IT	14	23	61	204
		CPU	0.0084	0.0283	0.2427	6.8279
		RES	$9.0191e - 8$	$9.9463e - 8$	$9.3622e - 8$	$9.7081e - 8$
$q = 100$	Picard-HSS	IT	14	20	64	65
		CPU	0.0151	0.0372	0.4811	4.2747
		RES	$9.8364e - 8$	$9.8128e - 8$	$9.4024e - 8$	$9.8372e - 8$
	Picard-SHSS	IT	83	44	71	63
		CPU	0.0508	0.0384	0.2824	2.2035
		RES	$9.9168e - 8$	$9.9270e - 8$	$9.8733e - 8$	$9.5253e - 8$

